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APPLICATION OF RECCURENT SEQUENCES

Abstract. *The article is devoted to different approaches of using the Fibonacci numbers and Lucas numbers. Some string problems were solved. Two important generalizations were obtained. Also was showed how applying limits and mathematical induction allows to prove significant corollaries.*

Keywords: *reccurent sequences; Fibonacci numbers; Lucas numbers; string problems; mathematical induction.*

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ЗАСТОСУВАННЯ РЕКУРЕНТНИХ ПОСЛІДОВНОСТЕЙ

Стаття присвячена різним підходам до використання чисел Фібоначчі та чисел Лукаса. Були розв'язані деякі задачі про рядки та розглянуто два важливих узагальнення. Також було показано як теорія границь та математична індукція дозволяють доводити різні важливі наслідки.

Ключові слова: *рекурентна послідовність; числа Фібоначчі; числа Лукаса; задачі з рядками; математична індукція.*

Introduction. The Fibonacci sequence and Fibonacci numbers are widely used in different branches of both mathematical and non-mathematical world. It is not surprising that the study of this issue continued intensively in the TWENTIETH century [1–2]. This was facilitated by new problems of combinatorics, informatics, which at that time faced the intellectual elite of society [2–3]. This topic does not lose its relevance to this day and Fibonacci numbers remains one of the most exciting sections of mathematics.

Considering the famous rabbit puzzle [1] which Fibonacci published in 1202 we obtain Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots \quad (1)$$

and has become one of the most famous sequences in mathematics.

"Golden spiral"[4] can also be seen in the works of nature. The distance between the leaves (or branches) on the trunk of the plants are approximately as Fibonacci numbers. Cells of pineapple create the same spiral sequence, that is 34 spirals in one direction and 55 in another.

Shells, snails' houses, starfishes, tulips, scales on a spruce cone and especially shellfish clam formed according to the same scheme, with each increment of a shell adds itself, another segment according to Fibonacci numbers.

The seeds in the sunflower basket are lined along the spirals, which are twisted both from left to right and from right to left. Sunflower seeds are placed by spirals in the Fibonacci sequence. An inflorescence of sunflower with 34 spirals one way and 55 to another.

Lucas sequences are used in probabilistic Lucas pseudoprime tests, which are part of the commonly used Baillie-PSW test. LUC is a public-key cryptosystem based on Lucas sequences that implements the analogs of ElGarnal (LUCELG), Diffie-Hellman (LUCDIF), and RSA(LUCRSA) [5–8].

Description of problems. To solve some string problems. To obtain two important generalizations of classical Fibonacci numbers. Applying limits and mathematical induction to prove some corollaries.

Main results. How many ways can one climb a staircase with *n* steps, taking one or two steps at a time?

Solution. Any single climb can be represented by a string of ones and twos which sum to *n*. We define a_n as the number of different strings that sum to *n*. In Table 1, we list the possible strings for the first five values of *n*. It appears that the a_n's from the beginning of the Fibonacci sequence.

To derive a relationship between a_n and the Fibonacci numbers, consider the set of strings that sum to *n*. This set may be divided into two nonoverlapping subsets: those strings that start with one and those strings that start with two. For the subset of strings that start with one, the remaining part of the string must sum to *n* – 1; for the subset of strings that start with two, the remaining part of the string must sum to *n* – 2. Therefore, the number of strings that sum to *n* is equal to the number of strings that sum to *n* – 1 plus the number of strings that sum to *n* – 2. The number of strings that sum to *n* – 1 is given by a_{n-1} and the number of strings that sum *n* – 2 is given by a_{n-2}, so that

$$a_n = a_{n-1} + a_{n-2}$$

And from the table we have $a_1 = 1 = F_2$ and $a_1 = 2 = F_3$, so that $a_n = F_{n+1}$ for all positive integers *n*.

Table 1

Strings of ones and twos that add up to *n*

n	strings	a_n
1	1	1
2	11, 2	2
3	111, 12, 21	3
4	1111, 112, 121, 211, 22	5
5	11111, 1112, 1121, 1211, 2111, 122, 212, 221	8

1. Consider a string consisting of the first *n* natural numbers, 123…*n*. For each number in the string, allow it to either stay fixed or change places with one of its neighbors. Define a_n to be the number of different strings that can be formed. Examples for the first four values of *n* are shown in Table 2. Prove that $a_n = F_{n+1}$.

Table 2

Strings of natural obtained by allowing a number to stay fixed or changed places with its neighbor

n	strings	a_n
1	1	1
2	12, 21	2
3	123, 132, 213	3
4	1234, 1243, 1324, 2134, 2143	5

Solution. Consider the set of different possible strings. This set may be divided into two nonoverlapping subsets: those strings start with one and those strings for which one and two are interchanged. For the former, the remaining *n* – 1 numbers can form a_{n-1} different strings. For the latter, the remaining *n* – 2 numbers may can form a_{n-2} different strings. The total number of different strings is therefore given by the Fibonacci recursion relation

$$a_n = a_{n-1} + a_{n-2.}$$

Together with $a_1 = 1 = F_2$ and $a_1 = 2 = F_3$, we obtain $a_n = F_{n+1}$.

2. Consider a problem like that above, but now allow the first 1 to change places with the last *n,* as if the string lies on a circle. Suppose n ≥ 3, and define b_n as the number of different

strings that can be formed. Show that $b_n = L_n$, where L_n is the *n*-th Lucas number which satisfied the quality

$$L_n = F_{n+1} + F_{n-1}.$$

Solution. Again, consider the set of different possible strings. This set may be divided into two nonoverlapping subsets: those strings for which he one and *n* are not interchanged, and those strings for which they are interchanged. For the former, the number of different strings is given by $a_n = F_{n+1}$. For the latter, the number of different strings is given by $a_{n-2} = F_{n-1}$. We therefore have

$$b_n = F_{n+1} + F_{n-1}.$$

The relation satisfied by b_n is the same as that satisfied by the *n*th Lucas number, so that $b_n = L_n$.

3. Prove that

$$F_n = \frac{1}{5}(L_{n-1} + L_{n+1}).$$

Solution. We have

$$\frac{1}{5}(L_{n-1} + L_{n+1}) = \frac{1}{5}\big((F_{n-2} + F_n) + (F_n + F_{n+2})\big) = \frac{1}{5}(F_{n-2} + 2F_n + F_n + F_{n-1}).$$

Using recursion relation, we obtain

$$\frac{1}{5}(F_{n-2} + 2F_n + F_n + F_{n-1}) = \frac{1}{5}(F_{n-2} + 3F_n + F_n + F_{n-1}) = F_n.$$

Consider two generalizations of Fibonacci numbers.

1. *The Fibonacci numbers can be extended to zero and negative indices using tlie relation* $F_n = F_{n+2} - F_{n+1}$. *Determine* F_0 *and find a general formula for* F_{-n} *in terms of* F_n. *Prove your result using mathematical induction.*

Proof. We calculate the first few terms.

$$F_0 = F_2 - F_1 = 0,$$
$$F_{-1} = F_1 - F_0 = 1,$$
$$F_{-2} = F_0 - F_{-1} = -1,$$
$$F_{-3} = F_{-1} - F_{-2} = 2,$$
$$F_{-4} = F_{-2} - F_{-3} = -3,$$
$$F_{-5} = F_{-3} - F_{-4} = 5,$$
$$F_{-6} = F_{-4} - F_{-5} = -8.$$

The correct relation appears to be

$$F_{-n} = (-1)^{n+1} F_n \qquad (2)$$

We now prove (2) by mathematical induction.

Base case: Our calculation above already shows that (2) is true for n = 1 and , n = 2, that is, $F_{-1} = F_1$ and $F_{-2} = -F_2$

Induction step: Suppose that (2) is true for positive integers $n = k$ -1 and n = *k*. Then using the definition, the induction hypothesis, and the recursion relation we have

$$F_{-(k+1)} = F_{-(k-1)} - F_{-k} = (-1)^k F_{k-1} - (-1)^{k+1} F_k =$$
$$= (-1)^{k+2}(F_{k-1} + F_k) = (-1)^{k+2} F_{k+1},$$

so that (2) is true for n= k+1. By the principle of induction, therefore (2) is true for all positive integers.

2. *The generalized Fibonacci sequence satisfies f_{n+1} = f_n + f_{n-1} with starting values f_1 = p and f_2 = q. Using mathematical induction, prove that*

$$f_{n+2} = F_{nP} + F_{n+1q}. \quad (3)$$

Proof. Prove (3) by mathematical induction.

Base case: To prove that (3) is true for n=1, we write

$$F_{1p} + F_{2q} = \mathrm{p} + \mathrm{q} = f_3.$$

To prove that (3) is true for *n* = 2, we write

$$F_{2p} + F_{3q} = \mathrm{p} + 2\mathrm{q} = f_3 + f_2 = f_4.$$

Induction step: Suppose that (3) is true for positive integers n=k-1 and n=k. Then using the induction hypothesis and the recursion relation we have

$$f_{k+3}\, f_{k+2} + f_{k+1} = (F_{kP} + F_{k+1q}) + (F_{k-1P}\, F_{kq}) =$$
$$= (F_k\, F_{k-1})\mathrm{p} + (F_{K+1} + F_k)\mathrm{q} = F_{k+1P} + F_{k+2q},$$

so that (2) is true for n=k+1. By the principle of induction, therefore (2) is true for all positive integers.

Consider another approach of using the Fibonacci numbers. The recursion relation for the Fibonacci numbers is given by

$$F_{n+1} = F_n + F_{n-1}$$

Diving by F_n yields

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}. \quad (4)$$

We assume that the ratio of two consecutive Fibonacci numbers approaches a limit as $n \to \infty$. Define $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$ so that $\lim\limits_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\alpha}$. Taking the limit, (4) becomes $\alpha = 1 + \frac{1}{\alpha}$, the same identity satisfied by the golden ratio. Therefore, if the limit exists, the ratio of two consecutive Fibonacci numbers must approach the golden ratio for large n, that is,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi = \frac{\sqrt{5}+1}{2}.$$

The ratio of consecutive Fibonacci numbers and this ratio minus the golden ratio is shown is Table 3. The last column appears to be approaching zero.

Table 3

Ratio of consecutive Fibonacci numbers approaches Φ

n	$\frac{F_{n-1}}{F_n}$	value	$\frac{F_{n+1}}{F_n} - \Phi$
1	$\frac{1}{1}$	1.0000	-0.6180
2	$\frac{2}{1}$	2.0000	0.3820

End of table 3

3	$\frac{3}{2}$	1.5000	-0.1180
4	$\frac{5}{3}$	1.6667	0.0486
5	$\frac{8}{5}$	1.6000	-0.0180
6	$\frac{13}{8}$	1.6250	0.0070
7	$\frac{21}{13}$	1.6154	-0.0026
8	$\frac{34}{21}$	1.6190	0.0010
9	$\frac{55}{34}$	1.6176	-0.0004
10	$\frac{89}{55}$	1.6782	0.0001

Prove some corollaries.

1. Assuming $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \Phi$, prove that

$$\lim_{k\to\infty}\frac{F_{k+n}}{F_k} = \Phi^n$$

Proof. Write

$$\frac{F_{k+n}}{F_k} = \frac{F_{k+n}}{F_{k+n-1}} \times \frac{F_{k+n-1}}{F_{k+n-2}} \times \ldots \times \frac{F_{k+1}}{F_k}.$$

Then taking $\underset{k\to\infty}{lim}$, and using

$$\lim_{j\to\infty}\frac{F_j}{F_{j-1}}\Phi,$$

One obtains directly

$$\lim_{k\to\infty}\frac{F_{k+n}}{F_k} = \Phi^n.$$

2. Using $\Phi^2 = \Phi + 1$, prove by mathematical inducting the following linearization of powers of the golden ratio:

(5)

Where n is a positive integer and $F_0 = 0$.

Proof. We prove (2) by mathematical induction.

Base case: For $n = 1$, the relation (5) becomes , which is true.

Induction step: Suppose that (5) is true for positive integer n = k. Using induction hypothesis, $\Phi^2 = \Phi + 1$ and recursion relation we have.

So that (2) is true for $n = k + 1$. By the principle of induction, (2) is therefore for all positive integers.

3. Using $\varphi^2 = -\varphi + 1$, prove by mathematical induction the following linearization of powers og the golden ratio conjugate:

$$(-\varphi)^n = -F_n\varphi + F_{n-1} \quad (6)$$

Where n is a positive integer and $F_0 = 0$.

Proof. We prove (3) by mathematical induction.

Base case: For $n = 1$, the relation (6) becomes $-\varphi = -\varphi$, which is true.

Induction step: Suppose that (6) is true for positive integer n = k. Using induction hypothesis, $\varphi^2 = -\varphi + 1$ and recursion relation, we have

$$(-\varphi)^{k+1} = -\varphi(-\varphi)^k = -\varphi(-F_k\varphi + F_{k-1}) = F_k\varphi^2 - F_{k-1}\varphi =$$
$$= F_k(-\varphi + 1) - F_{k-1}\varphi = -(F_k + F_{k-1})\varphi + F_k = -F_{k+1}\varphi + F_k.$$

So that (6) is true for $n = k + 1$. By the principle of induction, (2) is therefore for all positive integers.

Conclusions. Despite being invented more than 800 years ago Fibonacci numbers appear relevant and allow to obtain a lot of significant results.

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