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ON EXPONENTIAL DICHOTOMY FOR ABSTRACT DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENT

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We consider linear differential equations of the first order with delayed arguments in a Banach space. We establish conditions for the operator coefficients necessary for the existence of exponential dichotomy on the real axis. It is proved that the analyzed differential equation is equivalent to a difference equation in a certain space. It is shown that, under the conditions of existence and uniqueness of a solution bounded on the entire real axis, the condition of exponential dichotomy is also satisfied for any bounded known function. The explicit formula for projectors, which form the dichotomy, is found for the case of a single delay.

Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space, let $L(X)$ be a space of linear continuous operators in X. Consider the differential equation

$$x'(t) = \sum_{k=1}^{m} A_k x(t-k), \quad t \in \mathbb{R}, \tag{1}$$

where $\{A_k\colon 1 \le k \le m\} \subset L(X)$. We call a function $x \in C^\infty(\mathbb{R}, X)$ that satisfies Eq. (1) a solution of this equation.

For inhomogeneous equations of this type, conditions of existence and uniqueness of a solution bounded on the entire axis in the form of a bounded known function and a formula for it are known [1, 2]. For equations without delay, these conditions enable one to establish an exponential dichotomy. Indeed, for the equation

$$x'(t) = Ax(t) + y(t), \quad t \in \mathbb{R},$$

where $A \in L(X)$, the known condition for the existence and uniqueness of a bounded solution on the entire axis for an arbitrary bounded known function has the form

$$\sigma(A) \cap \{is \mid s \in \mathbb{R}\} = \varnothing.$$

Under this condition, the spectrum of the operator A consists of two parts and the corresponding projectors P_- and P_+ in the Riss decomposition [3] enable one to obtain an exponential dichotomy for the homogeneous equation

$$x'(t) = Ax(t), \quad t \in \mathbb{R},$$

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namely, there exist constants $K > 0$ and $\beta > 0$ such that

$$\|e^{At}P_-\| \le Ke^{-\beta t}, \quad t \ge 0,$$

$$\|e^{-At}P_+\| \le Ke^{-\beta t}, \quad t \ge 0.$$

However, for the equation with delay

$$x'(t) = Ax(t-1) + y(t), \quad t \in \mathbb{R},$$

the situation is more complicated. The condition for the existence and uniqueness of a bounded solution on the entire axis for an arbitrary bounded known function has the form

$$\sigma(A) \cap \{ise^{is} \mid s \in \mathbb{R}\} = \varnothing.$$

The curve in this formula is a double spiral that divides the complex plane into a countable number of bounded parts. However, the belonging of the spectrum to one of these parts does not guarantee a dichotomy.

In the case of delay, the problem of an exponential dichotomy is studied by many known scientists. It is worth noting the works by Hale [4, 5], Boichuk and Samoilenko [6, 7], Boichuk, Pokutnyi, and Zhuravlev [8, 9].

In particular, a generalized exponential dichotomy for differential equations with delayed argument in a finite-dimensional space is established and investigated in [4, 5]. For a certain class of equations including equations of the form (1), in a finite-dimensional case, the following statement is obtained:

Theorem 1 [5]. *For any $\mu \in \mathbb{R}$, there exist projectors P_-^μ and P_+^μ and constants $K_\mu > 0$ and $\gamma_2 > \gamma_1 > 0$ such that for a C^0-semigroup $\{T(t)\colon\ t \ge 0\}$ generated by Eq. (1), the condition of generalized exponential dichotomy*

$$\|T(t)P_-^\mu\| \le K_\mu e^{(\mu-\gamma_2)t}, \quad t \ge 0,$$

$$\|T(-t)P_+^\mu\| \le K_\mu e^{-(\mu-\gamma_1)t}, \quad t \ge 0,$$

is satisfied.

Note that a generalized dichotomy is weaker than a classical one.

In the present paper, we propose an approach that enables one to construct a difference equation equivalent to a differential equation. By using this result, we establish an exponential dichotomy for Eq. (1). For an equation with one delay, we give the explicit form of projectors generating a dichotomy.

Reduction of a Differential Equation to a Difference Equation

We introduce the spaces $Z_0 = C([0,1], X)$ with uniform norm $\|\cdot\|_0$ and $Z_1 = C^1([0,1], X)$ with norm $\|x\|_1 = \|x\|_0 + \|x'\|_0,\ x \in Z_1$. For functions defined on a segment, by derivatives at ends, we mean to the corresponding one-sided derivatives.

Lemma 1. *If a function $x \in C^{\infty}(\mathbb{R}, X)$ is a solution of Eq. (1), then the sequence of functions $\{x_n(t) = x(t+n),\ t \in [0,1] \colon n \in \mathbb{Z}\} \subset Z_1$ is a solution of the difference-integral equation*

$$x_{n+1}(t) = x_n(1) + \sum_{k=1}^{m} A_k \int_0^t x_{n+1-k}(s)ds, \quad t \in [0,1], \quad n \in \mathbb{Z}. \tag{2}$$

Conversely, if a certain sequence of functions $\{x_n \colon n \geq 1\} \subset Z_1$ is a solution of the difference-integral equation (2), then the equalities

$$x(t) = x_n(t-n), \quad t \in [n, n+1), \quad n \in \mathbb{Z},$$

give a function $x \in C^{\infty}(\mathbb{R}, X)$, which is a solution of the differential equation (1).

Proof. If x is a solution of the differential equation (1), then

$$x'(t+n+1) = \sum_{k=1}^{n} A_k x(t+n+1-k), \quad t \in [0,1], \quad n \in \mathbb{Z},$$

i.e.,

$$x'_{n+1}(t) = \sum_{k=1}^{n} A_k x_{n+1-k}(t), \quad t \in [0,1], \quad n \in \mathbb{Z}.$$

Integrating terms of the equation from 0 to t, we obtain

$$x_{n+1}(t) - x_{n+1}(0) = \sum_{k=1}^{n} A_k \int_0^t x_{n+1-k}(s)ds + \int_0^t y_{n+1}(s)ds, \quad t \in [0,1], \quad n \in \mathbb{Z}.$$

Taking into account that $x_{n+1}(0) = x(n+1) = x_n(1),\ n \in \mathbb{Z}$, we obtain the required equality.

Conversely, if equality (2) is true, then, for $t = 0$, we have

$$x_{n+1}(0) = x_n(1), \quad n \in \mathbb{Z},$$

hence, $x \in C(\mathbb{R}, X)$. Differentiating equality (2), we obtain equality (1) (for the left-hand and right-hand derivatives at integral points, we have values from different equations that coincide in view of continuity of the function x). In particular, we get $x \in C^{\infty}(\mathbb{R}, X)$.

We introduce operators $C_1, \ldots, C_m \in L(Z_1)$ acting by the rules

$$(C_1 x)(t) = x(1) + A_1 \int_0^t x(s)ds, \quad t \in [0,1],$$

$$(C_k x)(t) = A_k \int_0^t x(s)ds,\ t \in [0,1], \quad 2 \leq k \leq m.$$

Then the difference-integral equation (2) can be rewritten in the form

$$x_{n+1} = \sum_{k=1}^{m} C_k x_{n+1-k}, \quad n \in \mathbb{Z}. \tag{3}$$

Note that it is convenient to rewrite this difference equation in the form

$$u_{n+1} = C u_n, \quad n \in \mathbb{Z}, \tag{4}$$

in the space Z_1^m with norm $\|w\| = \sum_{k=1}^{m} \|w_k\|_1$, where

$$u_n = (x_n, x_{n-1}, \ldots, x_{n-m+1})^T,$$

$$C = \begin{pmatrix} C_1 & C_2 & C_3 & \ldots & C_{m-1} & C_m \\ I & O & O & \ldots & O & O \\ O & I & O & \ldots & O & O \\ O & O & I & \ldots & O & O \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ O & O & O & \ldots & I & O \end{pmatrix}.$$

We investigate the existence of an exponential dichotomy for this equation under the condition

$$0 \notin \sigma\big(itI - A_1 e^{-it} - A_2 e^{-2it} - \ldots - A_m e^{-mit}\big), \quad t \in \mathbb{R}, \tag{5}$$

which is the known necessary and sufficient condition for the existence of a unique bounded solution of the inhomogeneous equation corresponding to (1) [1].

Lemma 2. *If condition (5) is satisfied, then*

$$\sigma(C) \cap S = \varnothing,$$

where $S = \{z \in \mathbb{C} \colon |z| = 1\}$ is a unit circle in the complex plane.

Proof. Let $\lambda \in \mathbb{C} \backslash \sigma(C)$. This means that for an arbitrary $v \in Z_1^m$, there exists a unique $u \in Z_1^m$ such that

$$Cu - \lambda u = v.$$

By using the definition of the operator C, we obtain the equations

$$u_1(1) + \sum_{k=1}^{m} A_k \int_0^t u_k(s)ds - \lambda u_1(t) = v_1(t), \quad t \in [0, 1],$$

$$u_k(t) - \lambda u_{k+1}(t) = v_k(t),\ 1 \le k \le m-1, \quad t \in [0,1],$$

or, for $\lambda \ne 0$,

$$u_1(1) + \sum_{k=1}^{m} A_k \int_0^t \lambda^{1-k} u_1(s)ds - \lambda u_1(t) = w(t), \quad t \in [0,1],$$

$$w(t) = (Wv)(t) := v_1(t) + \sum_{k=2}^{m} A_k \int_0^t \sum_{p=2}^{k} \lambda^{p-k} v_p(s)ds, \quad t \in [0,1],$$

where the function $w \in Z_1$ can be arbitrary for the corresponding choice of v. This integral equation for $\lambda \ne 0$ is equivalent to the boundary-value problem for the differential equation

$$\sum_{k=1}^{m} A_k \lambda^{1-k} u_1(t) - \lambda u_1'(t) = w'(t), \quad t \in [0,1],$$

and the boundary condition $u_1(1) - \lambda u_1(0) = w(0)$.

We introduce the operator $A = A(\lambda) = \sum_{k=1}^{m} A_k \lambda^{-k}$. If there exists a continuous inverse operator to the operator $\frac{1}{\lambda} e^A - I$, then, by standard methods, we easily verify that a unique solution of the boundary-value problem is the function

$$u_1(t) = ((C - \lambda I)^{-1} v)_1(t) = -\frac{1}{\lambda} \int_0^t e^{A(t-s)} w'(s)ds$$

$$+ e^{At} \frac{1}{\lambda} \left(\frac{1}{\lambda} e^A - I \right)^{-1} \left(\frac{1}{\lambda} \int_0^1 e^{A(1-s)} w'(s)ds + w(0) \right), \quad t \in [0,1], \quad w = Wv \in Z_1. \tag{6}$$

In particular, this is true for $\lambda \in S$, i.e., $\lambda = e^{i\varphi},\ \varphi \in \mathbb{R}$, because by the Dunford theorem on map of spectrum, we have

$$\sigma\left(\frac{1}{\lambda} e^A - I \right) = \{ e^{\mu} - 1 \mid \mu \in \sigma(-i\varphi I + A_1 e^{-i\varphi} + A_2 e^{-2i\varphi} + \ldots + A_m e^{-mi\varphi}) \} \not\ni 0,$$

where we used that the condition $\mu \in \sigma(-i\varphi I + A_1 e^{-i\varphi} + A_2 e^{-2i\varphi} + \ldots + A_m e^{-mi\varphi})$ implies that

$$\mu - 2\pi k i \in \sigma(-i(\varphi + 2\pi k)I + A_1 e^{-i(\varphi+2\pi k)} + A_2 e^{-2i(\varphi+2\pi k)} + \ldots + A_m e^{-mi(\varphi+2\pi k)}), \quad k \in \mathbb{Z},$$

and, by the condition of the lemma, the last spectrum does not contain zero.

Exponential Dichotomy

For Eq. (2), consider the evolutionary operator $T(p),\ p\in\mathbb{Z}$, that for every $r\in\mathbb{Z}$ of the collection $\{x_k:\ r\le k\le r+m-1\}$ gives the collection $\{x_k:\ r+p\le k\le r+p+m-1\}$.

Theorem 2. *If condition (5) is satisfied, then the difference-integral equation (2) rewritten in the equivalent form (4) admits an exponential dichotomy: there exist subspaces Z_+ and Z_- of the space Z_1^m such that:*

(i) the direct sum of Z_- and Z_+ is equal to the space Z_1^m;

(ii) for the projector P_- onto the subspace Z_-, the estimate

$$\exists L>0\quad \exists q\in(0,1)\quad \forall p\ge 0:\ \ \|T(p)P_-\|_1\le Lq^p$$

is true;

(iii) for the projector P_+ onto the subspace Z_+, the estimate

$$\exists L>0\quad \exists q\in(0,1)\quad \forall p\ge 0:\ \ \|T(-p)P_+\|_1\le Lq^p$$

is true.

Proof. In view of Lemma 2, for Eq. (2) in the form (4), we can use the spectral Riss decomposition [3] and as Z_- and Z_+ we take subspaces corresponding to parts of the spectrum C lying inside and outside a unique circle, respectively. We obtain point 1. In addition,

$$\exists L>0\quad \exists q\in(0,1)\quad \forall p\ge 0:\ \ \|C^pP_-\|_1\le Lq^p,$$

$$\exists L>0\quad \exists q\in(0,1)\quad \forall p\ge 0:\ \ \|C^{-p}P_+\|_1\le Lq^p.$$

This yields estimates in points 2 and 3.

Corollary 1. *In the special case $m=1$ for the equation with one delay*

$$x'(t)=A_1x(t-1),\quad t\in\mathbb{R}, \tag{7}$$

the operator C coincides with the operator C_1 and acts in the space Z_1, condition (5) can be rewritten in the simpler form

$$\sigma(A_1)\cap\{ite^{it}\mid t\in\mathbb{R}\}=\varnothing,$$

the condition of application of dichotomy is belonging of the function x_1 to the subspace Z_- or Z_+, and projectors onto the subspaces Z_- and Z_+ are given, respectively, by the relations

$$(P_-x)(t)=x(t)+F(t)x(1)+\int_0^1 A_1F(t-s)x(s)ds,\quad t\in[0,1],$$

$$(P_+x)(t) = -F(t)x(1) - \int\limits_0^1 A_1 F(t-s)x(s)ds, \quad t \in [0,1],$$

where

$$F(t) = -\frac{1}{2\pi i}\int\limits_S \Big(\lambda e^{A_1\lambda} - I\Big)^{-1} e^{A_1\lambda t} d\lambda, \quad t \in \mathbb{R}.$$

Proof. To obtain these relations, recall that the projector onto the subspace Z_- has the form

$$P_- = -\frac{1}{2\pi i}\int\limits_S (C-\mu I)^{-1} d\mu = |\mu = 1/\lambda| = -\frac{1}{2\pi i}\int\limits_S \frac{1}{\lambda^2}\Big(C - \frac{1}{\lambda}I\Big)^{-1} d\lambda.$$

By using relation (6) and the notation $B(\lambda) = A(\lambda^{-1}) = A_1\lambda$, we obtain

$$\left(\frac{1}{\lambda^2}\Big(C - \frac{1}{\lambda}I\Big)^{-1} y\right)(t) = -\frac{1}{\lambda}\int\limits_0^t e^{B(\lambda)(t-s)} y'(s)ds$$

$$+ e^{B(\lambda)t}\frac{1}{\lambda}\Big(\lambda e^{B(\lambda)} - I\Big)^{-1}\left(\lambda\int\limits_0^1 e^{B(\lambda)(1-s)} y'(s)ds + y(0)\right),$$

$$t \in [0,1], \quad \lambda \in \mathbb{C}\backslash(\sigma(C) \cup \{0\}).$$

Integrating by parts, we get

$$\int\limits_0^t e^{B(\lambda)(t-s)} y'(s)ds = y(t) - e^{B(\lambda)t}y(0) + \int\limits_0^t B(\lambda)e^{B(\lambda)(t-s)}y(s)ds.$$

Hence,

$$\lambda\int\limits_0^1 e^{B(\lambda)(1-s)}y'(s)ds + y(0) = -\Big(\lambda e^{B(\lambda)} - I\Big)y(0) + \lambda y(1) + \lambda\int\limits_0^1 B(\lambda)e^{B(\lambda)(1-s)}y(s)ds$$

$$= -(\lambda e^{B(\lambda)} - I)y(0) + \lambda y(1) + (\lambda e^{B(\lambda)} - I)\int\limits_0^1 B(\lambda)e^{-B(\lambda)s}y(s)ds$$

$$+ \int\limits_0^1 B(\lambda)e^{-B(\lambda)s}y(s)ds.$$

Thus,

$$\left(\frac{1}{\lambda^2}\left(C-\frac{1}{\lambda}I\right)^{-1}y\right)(t)=-\frac{1}{\lambda}y(t)-\frac{1}{\lambda}\int\limits_0^t B(\lambda)e^{B(\lambda)(t-s)}y(s)ds$$

$$+e^{B(\lambda)t}\left(\lambda e^{B(\lambda)}-I\right)^{-1}\left(y(1)+\frac{1}{\lambda}\int\limits_0^1 B(\lambda)e^{-B(\lambda)s}y(s)ds\right)$$

$$+\frac{1}{\lambda}\int\limits_0^1 B(\lambda)e^{B(\lambda)(t-s)}y(s)ds.$$

We have

$$-\frac{1}{2\pi i}\int\limits_S\left(-\frac{1}{\lambda}I\right)d\lambda=I,\qquad -\frac{1}{2\pi i}\int\limits_S\left(-\frac{1}{\lambda}B(\lambda)e^{B(\lambda)(t-s)}\right)d\lambda=B(0)e^{B(0)(t-s)}=O.$$

In addition,

$$-\frac{1}{2\pi i}\int\limits_S\left(\frac{1}{\lambda}\left(\lambda e^{B(\lambda)}-I\right)^{-1}B(\lambda)e^{B(\lambda)(t-s)}\right)d\lambda=\sum_{k=1}^m A_kF_k(t-s),$$

where

$$F_k(t)=-\frac{1}{2\pi i}\int\limits_S\left(\lambda^{k-1}\left(\lambda e^{B(\lambda)}-I\right)^{-1}e^{B(\lambda)t}\right)d\lambda.$$

Since $F_1=F$ and

$$F_{k+1}(t)=-\frac{1}{2\pi i}\int\limits_S\left(\lambda^{k-1}\left(\lambda e^{B(\lambda)}-I\right)^{-1}e^{B(\lambda)(t-1)}\right)\left(\lambda e^{B(\lambda)}-I+I\right)d\lambda$$

$$=-\frac{1}{2\pi i}\int\limits_S\lambda^{k-1}e^{B(\lambda)(t-1)}d\lambda+F_k(t-1)=F_k(t-1),\quad t\in\mathbb{R},\quad 1\le k\le m-1,$$

we have $F_k(t)=F(t-k+1),\ t\in\mathbb{R},\ 1\le k\le m.$ By using the defintion of the functions F and $B,$ we get

$$(P_-y)(t)=y(t)+F(t)y(1)+\int\limits_0^1 A_1F(t-s)y(s)ds.$$

Corollary 2. *If x is a solution of the differential equation (7), then its components $x_+=P_+x$ and $x_-=P_-x$ are solutions of this equation that satisfy the estimates*

$$\exists K>0\quad\exists\beta>0\quad\forall t\ge 0:\ \|x_-(t)\|\le Ke^{-\beta t},$$

$$\exists K > 0 \quad \exists \beta > 0 \quad \forall t \le 0\colon\ \|x_+(t)\| \le Ke^{\beta t}.$$

Proof. Since the projectors P_- and P_+ commute with the operator $C = C_1$, applying them to both sides of Eq. (4), we show that x_- and x_+ are solutions. Estimates follow from the estimates of the theorem.

Application to the Inhomogeneous Equation

Consider the inhomogeneous differential equation

$$x'(t) = \sum_{k=1}^{m} A_k x(t-k) + y(t), \quad t \in \mathbb{R}, \tag{8}$$

where $\{A_k\colon 1 \le k \le m\} \subset L(X)$. If $y \in C(\mathbb{R}, X)$ is the known function, then the function $x \in C^1(\mathbb{R}, X)$ that satisfies Eq. (1) is called a solution of this equation.

To use estimates obtained by an exponential dichotomy, consider the inhomogeneous equation for (2):

$$x_{n+1}(t) = x_n(1) + \sum_{k=1}^{m} A_k \int_0^t x_{n+1-k}(s)ds + z_n(t), \quad t \in [0,1], \quad n \in \mathbb{Z}, \tag{9}$$

where $\{z_n\colon n \ge 1\} \subset Z_1$ is the known sequence and $\{x_n\colon n \ge 1\} \subset Z_1$ is the required sequence.

Lemma 3. *Suppose that for the given function $y \in C(\mathbb{R}, X)$, the function $x \in C^1(\mathbb{R}, X)$ is a solution of Eq. (8). Then the sequence of functions $\{x_n\colon n \in \mathbb{Z}\} \subset Z_1$ given by the equalities*

$$x_n(t) = x(t+n), \quad t \in [0,1], \quad n \in \mathbb{Z},$$

is a solution of the difference-integral equation (9) in which

$$z_n(t) = \int_0^t y(s+n+1)ds, \quad t \in [0,1], \quad n \in \mathbb{Z}$$

(moreover, $\{z_n\colon n \in \mathbb{Z}\} \subset Z_1$).

Proof is similar to the proof of Lemma 1.

By using this lemma, we reduce Eq. (8) to the difference-integral equation (9), which, in turn, can be rewritten in the form

$$x_{n+1} = \sum_{k=1}^{m} C_k x_{n+1-k} + z_n, \quad n \in \mathbb{Z}. \tag{10}$$

It is convenient to represent this difference equation in the form

$$u_{n+1} = Cu_n + v_n, \quad n \in \mathbb{Z}, \tag{11}$$

in the space Z_1^m with norm $\|u\| = \sum_{k=1}^{m} \|u_k\|_1$, where

$$u_n = (x_n, x_{n-1}, \ldots, x_{n-m+1})^T,$$

$$v_n = (z_n, 0, \ldots, 0)^T, \quad n \in \mathbb{Z}.$$

The existence, uniqueness, and estimates of solutions of abstract difference equations of this type are studied, in particular, in [10–12]. In particular, it is known that the existence and uniqueness of a bounded solution do not change in the passage to this equation [10, 11].

In the presence of a dichotomy, Eq. (8) can be split into two equations in the spaces Z_- and Z_+, which enables one to describe the asymptotic behavior at infinity [11].

In the case of one delay, a very simple result can be found. In this case, if x is a solution of the differential equation (7), then its components x_+ and x_- given by the formulas

$$x_+(t) = (P_+ x_n)(t-n), \quad t \in [n, n+1), \quad n \in \mathbb{Z},$$

$$x_-(t) = (P_- x_n)(t-n), \quad t \in [n, n+1), \quad n \in \mathbb{Z},$$

$$x_n(t) = x(t+n), \quad t \in [0,1], \quad n \in \mathbb{Z},$$

are solutions of the equations

$$x_-'(t) = A_1 x_-(t) + y_-(t), \quad t \in \mathbb{R},$$

$$x_+'(t) = A_1 x_+(t) + y_+(t), \quad t \in \mathbb{R},$$

where

$$y_+(t) = (P_+ y_n)(t-n), \quad t \in [n, n+1), \quad n \in \mathbb{Z},$$

$$y_-(t) = (P_- y_n)(t-n), \quad t \in [n, n+1), \quad n \in \mathbb{Z},$$

$$y_n(t) = y(t+n), \quad t \in [0,1], \quad n \in \mathbb{Z}.$$

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