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ON RESEARCH OF THE PROBLEM FOR THE INTEGRO-DIFFERENTIAL EQUATION

Summary: *The article offers an approach to establishment of conditions of existence and uniqueness of solution of the boundary value problem for the linear integro-differential equation with a parameter, as well as represents the method of reducing the given problem to the equivalent integral equation.*

The article is of theoretical nature. The results obtained may be used for treatment of specific applied problems. Theoretical computations are illustrated with specific example.

The work objective is to determine conditions for solvability of the boundary value problem for the linear integro-differential equation with a parameter.

The object of research is the boundary value problem for the linear integro-differential equation with a parameter.

The subject of research is conditions of existence and uniqueness of solution of the boundary value problem for the linear integro-differential equation with a parameter.

The research methods are basic methods of research and up-to-date results of the theory of differential and integral equations.

Keywords: boundary-value problem, integro-differential equation, uniqueness of a solution, integral equation.

The theory of differential, integral equations has recently gained widespread acceptance, which is explained, first of all, by relation of the given area with problems arising in mechanics, physics, biology, ecology, economics, etc. In addition to the application-oriented aspect being of prime importance, the mathematical part of the problems is also of some interest.

The issues of existence and uniqueness of solution were considered in the works of [1-7].

Researches, proving theorems of existence and uniqueness of solution of a posed problem was carried out based on methods developed in [2-4].

We consider the integro-differential equation

$$(Px)(t)=\int_a^b L(t,s)(Qx)(s)ds+f(t)+A(t)\mu \ . \tag{1}$$

We pose the problem of determination of a function $x\in W_2^m[a;b]$ and a parameter $\mu\in R^l$ such that satisfy the equation (1) almost everywhere, boundary condition and the limitation

$$B(x)=\delta, \tag{2}$$

$$\int_a^b Y(t)x(t)dt=\xi. \tag{3}$$

If such pair $(x(t),\mu)$ exists, then problem (1) - (3) is considered compatible.

Let us assume that:

1) $(Px)(t)=x^{(m)}(t)+p_1(t)x^{(m-1)}(t)+\ldots+p_m(t)x(t)$;

2) $(Qx)(t)=q_0x^r(t)+\ldots+q_r(t)x(t),\ \ r<m$;

3) the coefficients $\{p_1,\ldots,p_m,q_{0,\ldots,}q_r\}\subset L_2[a,b]$;

4) $t\in[a,b],\ \ f\in L_2[a,b]$;

5) the $(1\times l)-$matrix $A(t)$, the $(l\times 1)-$matrix $Y(t)$, whose elements are linearly independent functions square-summable on the interval $[a,b]$;

6) the constant $(m\times 1)-$matrix $B(t)$, whose elements take the form

$$B_\nu(x)\equiv\sum_{i=1}^{m}\left(\alpha_{\nu i}x^{(i-1)}(a)+\beta_{\nu i}x^{(i-1)}(b)\right);$$

7) $\delta\in R^m$, $\xi\in R^l$ – are given;

8) the kernel $L(t,s)$ – is square-summable in the totality of variables.

We will show that problem (1) - (3) is equivalent to an integral equation without limitations. Let us consider the generating problem

$$(Cx)(t)=A(t)\mu+y(t),\quad B(x)=\delta, \tag{4}$$

$$\int_a^b Y(t)x(t)dt=\xi, \tag{5}$$

where

$$(Cx)(t)\equiv x^{(m)}(t)+c_1(t)x^{(m-1)}(t)+\ldots+c_m(t)x(t), \tag{6}$$

in addition, the given function $y\in L_2[a,b]$, and the coefficients $c_1(t),\ldots,c_m(t)$ are continuous on the interval $[a,b]$.

Lemma. If the homogeneous problem

$$(Cx)(t)=A(t)\mu,\quad B(x)=0,\quad \int_a^b Y(t)x(t)dt=0 \tag{7}$$

has only the trivial solution, then there exist a vector $\sigma\in R^l$, functions $z(t)$, $G(t,s)$ and a $(m\times 1)-$matrix $T(s)$, such that the unique solution of the inhomogeneous problem (4), (5) is given by the formulas

$$x(t)=z(t)+\int_a^b G(t,s)y(s)ds, \tag{8}$$

$$\mu = \sigma + \int_a^b T(s)y(s)ds, \tag{9}$$

and the properties

$$\int_a^b T(s)A(s)ds = -I, \quad \int_a^b G(t,s)A(s)ds = 0, \tag{10}$$

where $I -$ is the identity matrix in R^l, are valid.

Indeed, let us assume that the coefficients $c_1(t),\dots,c_m(t)$ are selected so that there exists the Green function $\Gamma(t,s)$ of the problem

$$(Cx)(t) = w(s), \quad B(x) = \delta, \tag{11}$$

and it can be constructed explicitly. Under such assumption, the unique solution of problem (4) is given by the formula

$$x(t) = v(t) + \int_a^b \Gamma(t,s)(A(s)\mu + y(s))ds, \tag{12}$$

where $v(t) -$ is the solution of problem (11) for $w(s) = 0$. We write relation (12) as

$$x(t) = v(t) + N(t)\mu + \int_a^b \Gamma(t,s)y(s)ds, \tag{13}$$

where

$$N(t) = \int_a^b \Gamma(t,s)A(s)ds. \tag{14}$$

In order to determine the parameter μ we substitute relation (13) in condition (3)

$$\int_a^b Y(t)v(t)dt + \int_a^b Y(t)N(t)dt\mu + \int_a^b Y(t)\int_a^b \Gamma(t,s)y(s)dsdt = \xi.$$

After some transformations, we obtain the system of linear algebraic equations

$$\int_a^b Y(t)N(t)dt\mu = \xi - \int_a^b Y(t)v(t)dt - \int_a^b Y(t)\int_a^b \Gamma(t,s)y(s)dsdt\,. \qquad (15)$$

Let us denote in (15)

$$U = \int_a^b Y(t)x(t)dt\,, \quad \eta = \xi - \int_a^b Y(t)v(t)dt\,, \qquad (16)$$

$$D(s) = \int_a^b Y(t)\Gamma(t,s)dt\,. \qquad (17)$$

Then system (15) can be written as

$$U\mu = \eta - \int_a^b D(s)y(s)ds. \qquad (18)$$

We note that, under the condition of lemma, it is easy to prove the nonsingularity of the matrix U.

Having solved system (18), we obtain

$$\mu = U^{-1}\eta - \int_a^b U^{-1}D(s)y(s)ds. \qquad (19)$$

After the introduction of the notation

$$T(s) = -U^{-1}D(s), \quad \sigma = U^{-1}\eta, \qquad (20)$$

relation (19) takes the form (9), namely

$$\mu = \sigma + \int_a^b T(s)y(s)ds\,.$$

We now substitute relation (9) in (13) and after simple transformations, we obtain $x(t) = v(t) + N(t)\left(\sigma + \int_a^b T(s)y(s)ds\right) + \int_a^b \Gamma(t,s)y(s)ds =$

$$=v(t)+N(t)\sigma+\int_a^b N(t)T(s)y(s)ds+\int_a^b \Gamma(t,s)y(s)ds=$$

$$=v(t)+N(t)\sigma+\int_a^b (N(t)T(s)+\Gamma(t,s))y(s)ds. \qquad (21)$$

For convenience, we introduce the notation

$$z(t)=v(t)+N(t)\sigma, \qquad (22)$$

$$G(t,s)=N(t)T(s)+\Gamma(t,s) \qquad (23)$$

and then we obtain the formula (8)

$$x(t)=z(t)+\int_a^b G(t,s)y(s)ds.$$

We now verify the validity of relations (10), using designations (20), (17), (14), (16)

$$\int_a^b T(s)A(s)ds=-\int_a^b U^{-1}\int_a^b Y(t)\Gamma(t,s)dt\cdot A(s)ds=$$

$$=-U^{-1}\int_a^b Y(t)\int_a^b \Gamma(t,s)A(s)dsdt=-U^{-1}\int_a^b Y(t)N(t)dt=-U^{-1}\cdot U=-I.$$

The validity of the second relation in (10) is verified with the use of the same designations, formula (23), and the first relation in (10). We have

$$\int_a^b G(t,s)A(s)ds=\int_a^b (\Gamma(t,s)+N(t)T(s))A(s)ds=$$

$$=\int_a^b \Gamma(t,s)A(s)ds+\int_a^b N(t)T(s)A(s)ds=N(t)-N(t)=0.$$

Hence, the lemma is proved.

We can now reduce problem (1) - (3) to an integral equation. For this purpose, we write (1) in the form

$$x^{(m)}(t)+c_1(t)x^{(m-1)}(t)+\ldots+c_m(t)x(t)=$$

$$=x^{(m)}(t)+c_1(t)x^{(m-1)}(t)+\ldots+c_m(t)x(t)-$$

$$-x^{(m)}(t)-p_1(t)x^{(m-1)}(t)-\ldots-p_m(t)x(t)+$$

$$+\int_a^b L(t,s)\left(q_0(s)x^{(l)}(s)+\ldots+q_l(t)x(t)\right)ds+f(t)+A(t)\mu,$$

or in the form relation (4) as

$$(Cx)(t)=A(t)\mu+y(t),$$

by introducing the notation

$$h_k(t)=c_k(t)-p_k(t),\ \ k=\overline{1,m},$$

$$y(t)=\int_a^b L(t,s)\left(q_0(s)x^{(l)}(s)+\ldots+q_l(t)x(t)\right)ds+$$

$$+f(t)+h_1(t)x^{(m-1)}(t)+\ldots+h_m(t)x(t), \qquad (24)$$

and using formula (6).

Substituting the relation (8) in the right-hand side of relation (24), we obtain

$$y(t)=\int_a^b L(t,s)q_0(s)z^{(l)}(s)ds+\int_a^b\int_a^b L(t,\psi)q_0(\psi)\frac{\partial^l}{\partial\psi^l}G(\psi,s)y(s)d\psi\, ds+\ldots+$$

$$+\int_a^b L(t,s)q_l(s)z(s)ds+\int_a^b\int_a^b L(t,\psi)q_l(\psi)G(\psi,s)y(s)d\psi\, ds+$$

$$+f(t)+h_1(t)z^{(m-1)}(t)+\int_a^b h_1(s)\frac{\partial^{m-1}}{\partial t^{m-1}}G(t,s)y(s)ds+\ldots+$$

$$+h_m(t)z(t)+\int_a^b h_m(s)G(t,s)y(s)ds.$$

After some transformations, we obtain

$$y(t)=\int_a^b L(t,s)\left(q_0(s)z^{(l)}(s)+\ldots+q_l(s)z(s)\right)ds+$$

$$+\int_a^b\int_a^b L(t,\psi)\left(q_0(\psi)\frac{\partial^l}{\partial\psi^l}G(\psi,s)+\ldots+q_l(\psi)G(\psi,s)\right)y(s)d\psi\, ds+$$

$$+f(t)+h_1(t)z^{(m-1)}(t)+\ldots+h_m(t)z(t)+$$

$$+\int_a^b\left(h_1(t)\frac{\partial^{m-1}}{\partial t^{m-1}}G(t,s)+\ldots+h_m(t)\frac{\partial^{m-1}}{\partial t^{m-1}}G(t,s)\right)y(s)ds. \qquad (25)$$

Let us denote

$$K(t.s)=h_1(t)\frac{\partial^{m-1}}{\partial t^{m-1}}G(t,s)+\ldots+h_m(t)G(t,s)+$$

$$+\int_a^b L(t,\psi)\left(q_0(\psi)\frac{\partial^l}{\partial\psi^l}G(\psi,s)+\ldots+q_l(\psi)G(\psi,s)\right)d\psi, \qquad (26)$$

$$g(t)=f(t)+h_1(t)z^{(m-1)}(t)+\ldots+h_m(t)z(t)+$$

$$+\int_a^b L(t,s)\left(q_0(s)z^{(l)}(s)+\ldots+q_l(s)z(s)\right)ds, \qquad (27)$$

Then relation (25) takes the form

$$y(t)=g(t)+\int_a^b K(t,s)y(s)ds. \qquad (28)$$

The simple reasoning yields the validity of the following assertion: problem (1) - (3) is equivalent to the integral equation (28).

The equivalence is understood in the following meaning: if $y^*(t)$ is a solution of equation (28), then the function $x^*(t)$ and the parameter μ^* that are defined by the formulas

$$x^*(t) = z(t) + \int_a^b G(t,s)y^*(s)ds, \tag{29}$$

$$\mu^* = \sigma + \int_a^b T(s)y^*(s)ds, \tag{30}$$

Conversely, if $x^*(t)$ and μ^* are a solution of problem (1) - (3), then the function

$$y^*(t) = (Cx^*)(t) - A(t)\mu^* \tag{31}$$

– is a solution of equation (28).

Theorem. If the matrix U, given by formula (16) is nonsingular, then problem (1) - (3) is compatible if a solution of the integral equation (28) exists.

Example. We now reduce the boundary-value problem

$$x''(t) + 10\sqrt{t}\,x(t) = 5\int_0^1 \left(3\sqrt{ts} - 2\right)x(s)ds + 10t^3 + 3\mu, \tag{32}$$

$$x(0) = 1, \quad x(1) = 2, \tag{33}$$

$$\int_0^1 (7 - 9t)x(t)dt = \frac{5}{2} \tag{34}$$

to an integral equation of the form (28).

For this purpose, we consider the auxiliary problem

$$x''(t) = 3\mu + y(t), \quad x(0) = 1, \quad x(1) = 2, \tag{35}$$

$$\int_0^1 (7-9t)x(t)dt = \frac{5}{2} \tag{36}$$

and will construct its solution.

Having solved problem (35), we obtain

$$x(t) = 1 + t + \frac{3}{2}\left(t^2 - t\right)\mu + \int_0^1 \Gamma(t,s)y(s)ds, \tag{37}$$

$$\Gamma(t,s) = \begin{cases} t(s-1), & t \le s, \\ s(t-1), & t \ge s. \end{cases} \tag{38}$$

In order to determine the parameter μ, we substitute relation (37) in condition (36):

$$\int_0^1 (7-9t)\left(1 + t + \frac{3}{2}\left(t^2 - t\right)\mu + \int_0^1 \Gamma(t,s)y(s)ds\right)dt = \frac{5}{2}.$$

Making calculations with regard for (38), we obtain

$$\int_0^1 (7-9t)\left(1 + t + \frac{3}{2}\left(t^2 - t\right)\mu\right)dt = 3 - \frac{5}{8}\mu,$$

$$\int_0^1 (7-9t)\Gamma(t,s)dt = -\frac{3}{2}s^3 + \frac{7}{2}s^2 - 2s.$$

From whence, we have

$$3 - \frac{5}{8}\mu + \int_0^1 \left(-\frac{3}{2}s^3 + \frac{7}{2}s^2 - 2s\right)y(s)ds = \frac{5}{2},$$

and

$$\mu = \frac{4}{5} + \int_0^1 \left(-\frac{12}{5}s^3 + \frac{28}{5}s^2 - \frac{16}{5}s\right)y(s)ds. \tag{39}$$

Let us denote in (39)

$$\sigma = \frac{4}{5}, \quad T(s) = -\frac{12}{5}s^3 + \frac{28}{5}s^2 - \frac{16}{5}s, \tag{40}$$

then we obtain formula (9), namely

$$\mu = \sigma + \int_a^b T(s)y(s)ds.$$

Relations (37) and (39) yield

$$x(t) = 1 + t + \frac{3}{2}\left(t^2 - t\right)\left(\frac{4}{5} + \int_0^1 \left(-\frac{12}{5}s^3 + \frac{28}{5}s^2 - \frac{16}{5}s\right)y(s)ds\right) + \int_0^1 \Gamma(t,s)y(s)ds$$

or

$$x(t) = 1 - \frac{1}{5}t + \frac{6}{5}t^2 + \int_0^1 \left(\left(t^2 - t\right)\left(-\frac{18}{5}s^3 + \frac{42}{5}s^2 - \frac{24}{5}s\right) + \Gamma(t,s)\right)y(s)ds. \tag{41}$$

Hence, if we denote in (41)

$$z(t) = 1 - \frac{1}{5}t + \frac{6}{5}t^2, \tag{42}$$

$$G(t,s) = \left(t^2 - t\right)\left(-\frac{18}{5}s^3 + \frac{42}{5}s^2 - \frac{24}{5}s\right) + \Gamma(t,s), \tag{43}$$

we obtain formula (8), namely

$$x(t) = z(t) + \int_a^b G(t,s)y(s)ds.$$

We note that it is easy to verify by direct calculations with the use of formulas (40), (42), (43) that relations (10) hold. Indeed,

$$\int_a^b T(s)A(s)ds = \int_0^1 \left(-\frac{12}{5}s^3 + \frac{28}{5}s^2 - \frac{16}{5}s\right)\cdot 3 = -1,$$

$$\int_a^b G(t,s)A(s)ds = \int_0^1 \left(\left(t^2 - t\right)\left(-\frac{18}{5}s^3 + \frac{42}{5}s^2 - \frac{24}{5}s\right) + \Gamma(t,s) \right) \cdot 3ds =$$

$$= -\frac{3}{2}\left(t^2 - t\right) + \frac{3}{2}\left(t^3 - t^2\right) - \frac{3}{2}t^3 + 3t^2 - \frac{3}{2}t = 0.$$

In our case, formula (25) takes the form

$$y(t) = 5\int_0^1 \left(3\sqrt{ts} - 2\right)x(s)ds - 10\sqrt{t}\,x(t) + 10t^3. \qquad (44)$$

Let us substitute representation (41) in (44):

$$y(t) = 5\int_0^1 \left(3\sqrt{ts} - 2\right)x(s)ds - 10\sqrt{t}\,x(t) + 10t^3 =$$

$$= 5\int_0^1 \left(3\sqrt{ts} - 2\right)\left(1 - \frac{1}{5}s + \frac{6}{5}s^2\right)ds +$$

$$+ 5\int_0^1 \left(3\sqrt{ts} - 2\right)\int_0^1 \left(\left(\psi^2 - \psi\right)\left(-\frac{18}{5}s^3 + \frac{42}{5}s^2 - \frac{24}{5}s\right) + \Gamma(\psi,s) \right) y(s)d\psi\, ds + 10t^3 -$$

$$-10\sqrt{t}\left(1 - \frac{1}{5}t + \frac{6}{5}t^2\right) - 10\sqrt{t}\int_0^1 \left(\left(t^2 - t\right)\left(-\frac{18}{5}s^3 + \frac{42}{5}s^2 - \frac{24}{5}s\right) + \Gamma(t,s) \right) y(s)ds.$$

With regard for designations (42) and (43), we have

$$y(t) = 5\int_0^1 \left(3\sqrt{ts} - 2\right)z(s)ds + 5\int_0^1 \left(3\sqrt{ts} - 2\right)\int_0^1 G(\psi,s)y(s)d\psi\, ds + 10t^3 -$$

$$-10\sqrt{t}\,z(t) - 10\sqrt{t}\int_0^1 G(t,s)y(s)ds. \qquad (45)$$

In relation (45), we denote

$$g(t)=5\int_0^1\left(3\sqrt{ts}-2\right)z(s)ds-10\sqrt{t}\,z(t)+10t^3,$$

$$K(t,s)=5\int_0^1\left(3\sqrt{ts}-2\right)G(\psi,s)d\psi-10\sqrt{t}\,G(t,s),$$

then we obtain an integral equation of the form (28), i.e.,

$$y(t)=g(t)+\int_a^b K(t,s)y(s)ds.$$

Thus, problem (32) - (34) is reduced to the integral equation (45).

Hence, according to Theorem, problem (32) - (34) is compatible, and its solution calculated by formulas (39) and (41) takes the form

$$x^*(t)=1+t^2\sqrt{t}, \qquad \mu^*=4,(285714).$$

Conclusions of the given research and prospects for further researches. Based on the results obtained for integral equations it is established the conditions of compatibility of the indicated problem. A new approach to the study of the boundary-value problem for an integro-differential equation with parameters and restrictions by means of its reduction to an equivalent integral equation is proposed.

Thereafter it is planned to highlight and substantiate efficiency and conditions of application to the given problem of approximate methods, in particular, iterative and projective methods.

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